Proofs by example

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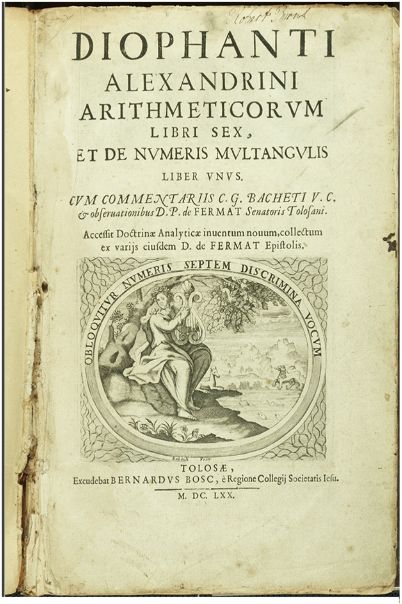

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Example: $1, 1, 2, 3, 5, 8, 13, \ldots$?

Another example: **Thales' theorem**

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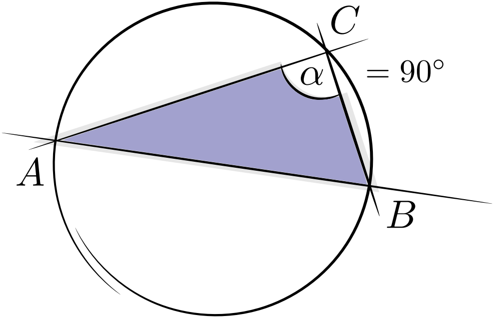

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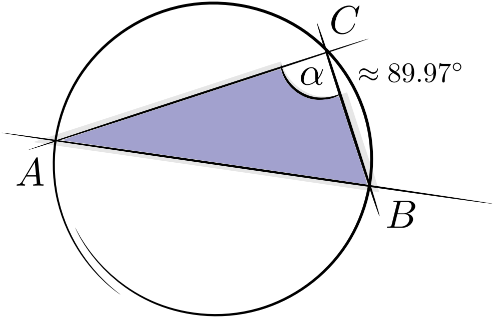

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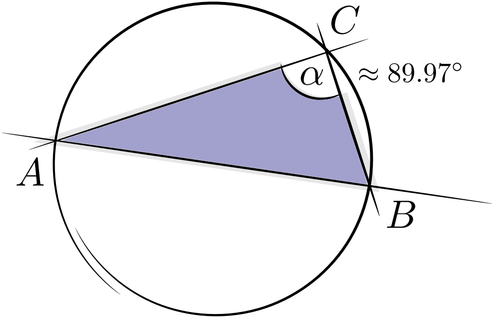

$\rightsquigarrow$ Can "Proof by example" work?

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Algebraic setting

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Schwartz-Zippel lemma (1979–80; Ore 1922):
If $A \subset \mathbb{C}$ finite, $p_1, \dots, p_n$ independent and uniformly at random from A, then

$$g \neq 0 \quad \Longrightarrow \quad P\big[g(p_1, \dots, p_n) = 0\big] \leq \frac{\deg g}{|A|}.$$

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Combinatorial Nullstellensatz (Alon 1999, weak):
If $A \subset \mathbb{C}$, $|A| > \deg g$, then

$$g(A \times \ldots \times A) = 0 \quad \Longrightarrow \quad g = 0.$$

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Lagrange's theorem (1798):
If $g(t) = a_0 + a_1 t + \ldots + a_{n-1} t^{n-1} + t^n$, then

$$|x| > \max\left(1, \sum |a_i|\right) \quad \Longrightarrow \quad g(x) \neq 0.$$

Proofs by example

Want:

- sufficiently generic example P,
- example P easy to construct,
- $g(P)$ easy to compute,
- allow for numerical margin of error.

Main theorem (over $\mathbb{Q}$ with standard $|\,.\,|$ (2019)).
Let

- $X = V(f_1, \ldots, f_m) \subseteq \overline{\mathbb{Q}}^n$ irreducible, $\dim X = d$,
- g polynomial,
- $H :=$ "arithmetic complexity" of $(f_1, \ldots, f_m, g)$,
- $P = (p_1, \ldots, p_n) \in \mathbb{Q}^n$ such that

$$0 \ll_H h(p_1) \ll_H h(p_2) \ll_H \ldots \ll_H h(p_d).$$

Let $\varepsilon := \varepsilon(H, h(p_d))$. Then

$$\text{if } \left\{ \begin{array}{l} |f_i(P)| \leq \varepsilon \ \ \forall i \ \text{ and} \\ |g(P)| \leq \varepsilon \end{array} \right\} \implies g|_X = 0.$$

Remarks

- "Robust one-point Nullstellensatz"
- Based on
  - arithmetic Bézout theorem [Bost–Gillet–Soulé (1991,94), Philippon]
  - arithmetic Nullstellensatz [Krick–Pardo–Sombra]
  - new effective Łojasiewicz inequality
- Way to remove irreducibility assumption on X.
- Way to remove knowledge of dimension of X.
- Motivates other "robust Nullstellensätze".
- Motivates more general combinatorial Nullstellensätze.

A comparison:

Let $X = V(f_1, \ldots, f_m)$.

Hilbert's Nullstellensatz:
$g|_X = 0 \iff g^N = \sum_i \lambda_i f_i$ for some N and some polynomials λ_i

Proof by example scheme:
$g|_X = 0 \iff g(P) \approx 0$ for some sufficiently generic P close to X

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$\rightsquigarrow$ new **witness** for $g|_X = 0$.

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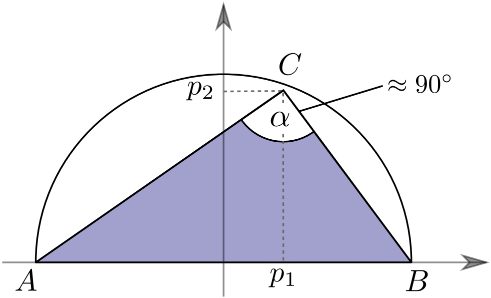

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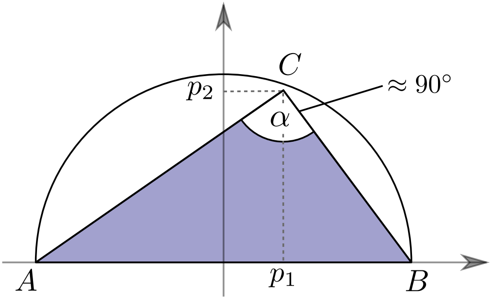

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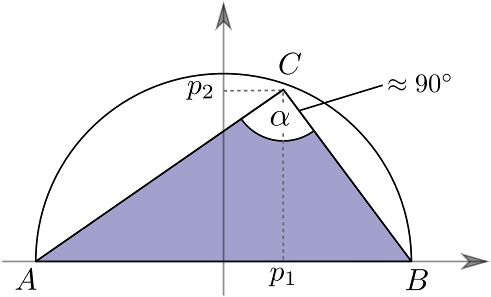

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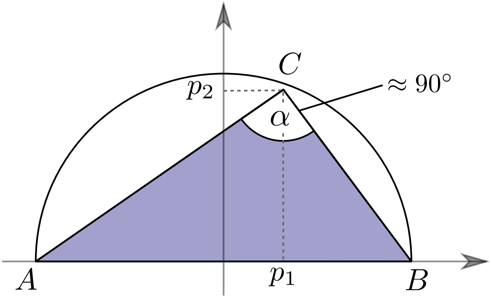

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$\rightsquigarrow$ works! $\square$

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If

- P sufficiently generic and close to X, and
- $|\det([e_1, e_2, \ldots, e_d, \nabla f_1(P), \ldots, \nabla f_{n-d}(P)])| > \varepsilon$,

then $\dim X = d$.

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then $\dim X = d$.

Note: ε is mild.

Equivalence if X is smooth.

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$\rightsquigarrow$ **Dichotomy theorem**:

If P sufficiently generic and close enough to X, then either

Case 1: $|g(P)| \leq \varepsilon$ and $g|_X = 0$.

Case 2: $|g(P)| \geq 2\varepsilon$ and $g|_X \neq 0$.

Proofs by example

Future topics:

1. Better bounds
2. Equivalence to arithmetic Nullstellensatz
3. Combinatorial Nullstellensatz for varieties
   $\rightsquigarrow$ Proofs by examples (e.g. Thales, Pappus, Desargues)
   $\rightsquigarrow$ Robust combinatorial/probabilistic Nullstellensätze
4. Comparison with Gröbner bases
5. Continuation of sequences

Thank you